Jacobians and complexity of I-graphs

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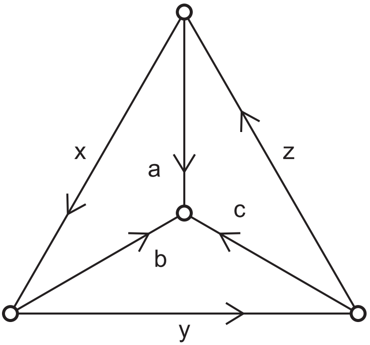
The notion of the Jacobian group of graph (also known as the Picard group, critical group, sandpile group, dollar group) was independently given by many authors (R. Cori and D. Rossin, M. Baker and S. Norine, N. L. Biggs, R. Bacher, P. de la Harpe and T. Nagnibeda). This is an important algebraic invariant of a finite graph. In particular, the order of the Jacobian group coincides with the number of spanning trees for a graph. The latter number is known for many large families of graphs. Still it usually difficult to find systematic description for the structure of Jacobian for such families. The aim of the present presentation provide structure theorems for Jacobians of I-graphs.

We define Jacobian $Jac(G)$ of a graph G as the Abelian group generated by flows satisfying the first and the second Kirchhoff laws. We illustrate this notion on the following simple example.

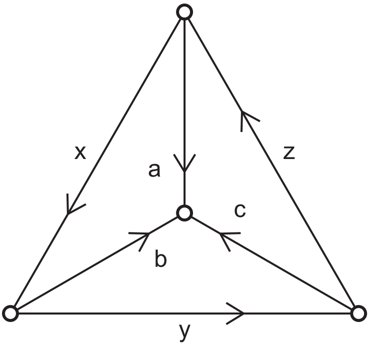
Jacobians



Complete graph K_4

The first Kirchhoff law is given by the equations

$$L_1 : \quad \begin{cases} a + b + c = 0; \\ x - y - b = 0; \\ y - z - c = 0; \\ z - x - a = 0. \end{cases}$$



Complete graph K_4

The second Kirchhoff law is given by the equations

$$L_2 : \quad \begin{cases} x + b - a = 0; \\ y + c - b = 0; \\ z + a - c = 0. \end{cases}$$

Jacobians

Now $Jac(K_4) = \langle a,\ b,\ c,\ x,\ y,\ z : L_1,\ L_2 \rangle$.
Since by $L_2 : x = a - b,\ y = b - c,\ z = c - a$ we obtain

$$\langle a,\ b,\ c : a{+}b{+}c = 0, a{+}b{+}c{-}4b = 0, a{+}b{+}c{-}4c = 0, a{+}b{+}c{-}4a = 0 \rangle =$$

$$\langle a,\ b,\ c : a + b + c = 0,\ 4a = 0,\ 4b = 0,\ 4c = 0 \rangle =$$

$$\langle a,\ b : 4a = 0,\ 4b = 0 \rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4.$$

So we have $Jac(K_4) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Basic Facts

The graphs under consideration are supposed to be finite and connected. They may have loops or multiple edges.
Let a_{uv} be the number of edges between two given vertices u and v of G.
The matrix $A = A(G) = [a_{uv}]_{u,v \in V(G)}$, is called the *adjacency matrix* of the graph G.
Let $d(v)$ denote the degree of $v \in V(G)$, $d(v) = \sum_u a_{uv}$, and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$.
The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G.
It should be noted that loops have no influence on $L(G)$. The matrix $L(G)$ is sometimes called the *Kirchhoff matrix* of G.

Basic Facts

Recall a wide known theorem about the structure of an arbitrary Abelian group.

Let $\mathcal{A}$ be a finite Abelian group generated by $x_1, x_2, \ldots, x_n$ and satisfying the system of relations

$$\sum_{j=1}^{n} a_{ij}x_j = 0,\ i = 1,\ \ldots,\ m,$$

where $A = \{a_{ij}\}$ is an integer $m \times n$ matrix. Set $d_j,\ j = 1,\ \ldots,\ r$, for the greatest common divisor of all $j \times j$ minors of A. Then,

$$\mathcal{A} \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2/d_1} \oplus \mathbb{Z}_{d_3/d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r/d_{r-1}}.$$

This is so called the Smith Normal Form of group $\mathcal{A}$.

Laplacian and Jacobians

Consider the Laplacian matrix $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$, where $|V| = |V(G)|$ is the number of vertices of G. Then $\mathrm{coker}(L(G)) = \mathbb{Z}^{|V|}/\mathrm{im(L(G))}$ is an abelian group. Let

$$\mathrm{coker}(L(G)) \cong \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_{|V|}},$$

be its Smith normal form satisfying $t_i \big| t_{i+1}$, $(1 \leq i \leq |V|)$. If graph G is connected then the groups $\mathbb{Z}_{t_1}, \mathbb{Z}_{t_1}, \ldots \mathbb{Z}_{t_{|V|-1}}$ are finite and $\mathbb{Z}_{t_{|V|}} = \mathbb{Z}$. In this case,

$$Jac(G) = \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_{|V|-1}}$$

is the Jacobian group of the graph G.
Equivalently $\mathrm{coker}(L(G)) \cong Jac(G) \oplus \mathbb{Z}$.

That is, $Jac(G)$ is the torsion subgroup of cokernel of $L(G)$.

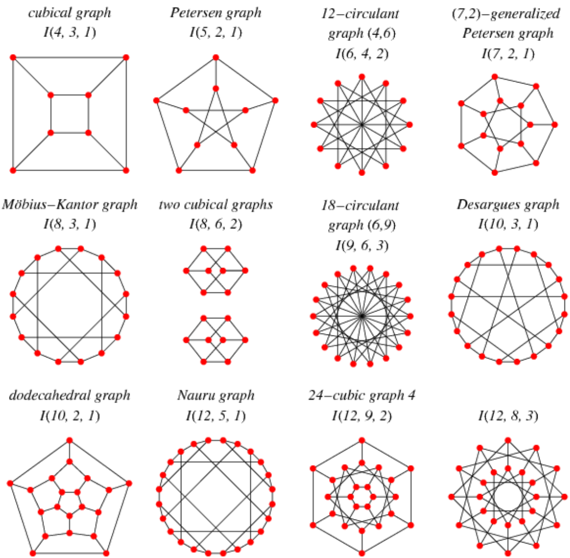
I-Graphs

I-graphs can be described in the following way:

$$\begin{aligned} V(I(n,k,l)) &= \{u_i, v_i \mid i = 1, 2, \dots, n\} \\ E(I(n,k,l)) &= \{u_i u_{i+l},\ u_i v_i,\ v_i v_{i+k} \mid i = 1, 2, \dots, n\}. \end{aligned}$$

where all subscripts are given modulo n.
Since $I(n,k,l) = I(n,l,k)$ we will usually assume that $k \leq l$. Here we will deal with 3-valent graphs only. This means that in the case of even n and $l = n/2$ the graph under consideration has multiple edges. The graph $I(n,l,k)$ is connected if and only if $GCD(n,k,l) = 1$. If $GCD(n,k,l) = m > 1$, then $I(n,k,l)$ is a union of m copies of the graph $I(n/m, k/m, l/m)$. If $m = 1$ and $GCD(k,l) = d$, then the graphs $I(n,k,l)$ and $I(n, k/d, l/d)$ are isomorphic.
The general properties of I-graphs were investigated by Tomo Pisanski and his collaborators.

I-graphs



cubical graph
I(4, 3, 1)
Petersen graph
I(5, 2, 1)
12−circulant
graph (4,6)
I(6, 4, 2)
(7,2)−generalized
Petersen graph
I(7, 2, 1)
Möbius−Kantor graph
I(8, 3, 1)
two cubical graphs
I(8, 6, 2)
18−circulant
graph (6,9)
I(9, 6, 3)
Desargues graph
I(10, 3, 1)
dodecahedral graph
I(10, 2, 1)
Nauru graph
I(12, 5, 1)
24−cubic graph 4
I(12, 9, 2)
I(12, 8, 3)

I-Graphs and its associated polynomial

For any given graph $I(n, k, l)$ we consider a Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$. The extended form of the polynomial $P(z)$ is $z^{-k-l} + a_1 z^{-k-l+1} + \ldots + a_{2k+2l-1} z^{k+l-1} + z^{k+l}$ where $a_1, a_2, \ldots, a_{2k+2l-1}$ are integer coefficients.

The following matrix $\mathcal{A}$ is called a *companion matrix* of the polynomial $P(z)$:

$$\mathcal{A} = \left(\begin{array}{c|c} 0 & I_{2k+2l-1} \\ \hline \multicolumn{2}{c}{-1, -a_1, \ldots, -a_{2k+2l-1}} \end{array} \right).$$

Here I_m is the identity matrix of the size m.

Jacobian of I-Graphs

The structure of Jacobian of a graph $I(n, k, l)$ is given by the following theorem.

Theorem (M.(2017))

The Jacobian group $Jac(I(n, k, l))$ of a connected I-graph $I(n, k, l)$ is isomorphic to the torsion subgroup of $coker(\mathcal{A}^n - I)$, where $\mathcal{A}$ is the companion matrix for the Laurent polynomial $(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$.

This theorem gives a simple way to find Jacobian group $\mathrm{Jac}(I(n, k, l))$ for small values of k, l and sufficiently large numbers n.

Jacobian of I-Graphs

The following result is the first consequence of the previous theorem.

Theorem

For any given I-graph $I(n,k,l)$ the minimum number of generators for Jacobian $Jac(I(n,k,l))$ is at least 2 and at most $2k+2l-1$.

For graphs $I(4,2,3)$ and $I(6,3,4)$, the Jacobian group $Jac(I(n,k,l))$ is generated by 2 elements. The upper bound $2k+2l-1$ for the minimum number of generators of $Jac(I(n,k,l))$ is attained for graph $I(34,2,3)$ and $I(170,3,4)$.
So, the lower bound 2 and the upper bound $2k+2l-1$ for the minimum number of generators of $Jac(I(n,k,l))$ are sharp.

Tables

n	$\mathrm{Jac}(I(n,2,3))$	$\tau_{2,3}(n) = \lvert\mathrm{Jac}(I(n,2,3))\rvert$
4	$\mathbb{Z}_7 \oplus \mathbb{Z}_{28}$	196
5	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{95}$	1805
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{83} \oplus \mathbb{Z}_{581}$	48223
8	$\mathbb{Z}_{161} \oplus \mathbb{Z}_{1288}$	207368
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_{1558} \oplus \mathbb{Z}_{3895}$	6068410
11	$\mathbb{Z}_{1693} \oplus \mathbb{Z}_{18623}$	31528739
12	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{665} \oplus \mathbb{Z}_{7980}$	132667500
13	$\mathbb{Z}_{25} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325}$	858203125
14	$\mathbb{Z}_{17513} \oplus \mathbb{Z}_{245182}$	4293872366
15	$\mathbb{Z}_{37069} \oplus \mathbb{Z}_{556035}$	20611661415
16	$\mathbb{Z}_{84847} \oplus \mathbb{Z}_{1357552}$	115184214544
17	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_{23186} \oplus \mathbb{Z}_{394162}$	584898568448

Tables

n	$\mathrm{Jac}(I(n,3,4))$	$\tau_{3,4}(n) = \lvert\mathrm{Jac}(I(n,3,4))\rvert$
5	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	2000
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{71} \oplus \mathbb{Z}_{497}$	35287
8	$\mathbb{Z}_{73} \oplus \mathbb{Z}_{584}$	42632
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60}$	5184000
11	$\mathbb{Z}_{1541} \oplus \mathbb{Z}_{16951}$	26121491
12	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{209} \oplus \mathbb{Z}_{2508}$	63424812
13	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{1555} \oplus \mathbb{Z}_{20215}$	785858125

The first example of Jacobian $Jac(I(n,3,4))$ with the maximum rank 13:

$$n = 170,$$

$$Jac(I(170,3,4)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4^8 \oplus \mathbb{Z}_{6108} \oplus \mathbb{Z}_{30540}$$

$$\oplus \mathbb{Z}_{2^2\cdot 3\cdot 5\cdot 103\cdot 509\cdot 1699\cdot 11593\cdot p\cdot q} \oplus \mathbb{Z}_{2^2\cdot 3\cdot 5\cdot 17\cdot 103\cdot 509\cdot 1699\cdot 11593\cdot p\cdot q},$$

where $p = 16901365279286026289$ and $q = 34652587005966540929$.

The number of spanning trees

Recall that the order of the Jacobian coincides with the number of spanning trees of a graph. We have

Theorem

The number of spanning trees of the I-graph $I(n, k, l)$ is given by the formula

$$\tau_{k,l}(n) = (-1)^{(n-1)(k+l)} n \prod_{s=1}^{k+l-1} \frac{T_n(w_s) - 1}{w_s - 1},$$

where $w_s, s = 1, 2, \ldots, k + l - 1$ are roots of the order $k + l - 1$ algebraic equation

$$\frac{(3 - 2T_k(w))(3 - 2T_l(w)) - 1}{w - 1} = 0,$$

and $T_n(w) = \cos(\arccos(n\, w))$ is the Chebyshev polynomial of the first kind.

Asymptotic for the number of spanning trees

The asymptotic for the number of spanning trees of the graph $I(n, k, l)$ is given in the following theorem.

Theorem

Let $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$. *Suppose that* k *and* l *are relatively prime and set* $A = \prod_{P(z)=0,\, |z|>1} |z|$. *Then the number* $\tau_{k,l}(n)$ *of spanning trees of the graph* $I(n, k, l)$ *has the asymptotic*

$$\tau_{k,l}(n) \sim \frac{n}{k^2 + l^2} A^n, \ n \to \infty.$$

We note that the number A coincides with the Mahler measure of Laurent polynomial $P(z)$. It gives a simple way to evaluate A using the following formula

$$A = \exp\Big(\int_0^1 \log |P(e^{2\pi i t})| \mathrm{d}t\Big).$$

Examples

1° The Prism graph $I(n,1,1)$. We have the following asymptotic $\tau_{1,1}(n) = n(T_n(2)-1) \sim \frac{n}{2}(2+\sqrt{3})^n$, $n \to \infty$.

2° The generalized Petersen graph $GP(n,2) = I(n,1,2)$. The the number of spanning trees behaves like $\tau_{1,2}(n) \sim \frac{n}{5}A_{1,2}^n$, $n \to \infty$, where $A_{1,2} = \frac{7+\sqrt{5}+\sqrt{38+14\sqrt{5}}}{4} \cong 4.39026$.

3° The smallest proper I-graph $I(n,2,3)$ has the following asymptotic for the number of spanning trees $\tau_{2,3}(n) \sim \frac{n}{13}A_{2,3}^n$, $n \to \infty$. Here $A_{2,3} \cong 4.84199$ is a root of an order 16 algebraic equation.